MATH1520C University Mathematics for Applications **Fall 2021**

Chapter 1: Notation and Functions

Learning Objectives:
(1) Identify the domain of a function, and evaluate a function from an equation.
(2) Gain familiarity with piecewise functions.
(3) Study the vertical line test.
(4) Know how to form and use composite functions.

1.1 Set

$1 \in \{1,2,3\}$

- **Set** is a collection of objects (called **elements**)

  $\{x \mid x^2 = 4\}$

  1. Order of elements does not matter. E.g. $\{1, 2, 3\} = \{3, 2, 1\}$.
  2. Representation of a set is not unique. E.g. $\{-2, 2\} = \{x \mid x^2 = 4\}$.

- $\in$: belongs to. If a is an element of A, we say that a belongs to A; denoted as $a \in A$.
- $\subset$: subset of. Let A, B be two sets such that $\forall a \in A$, $a \in B$. Then we say that A is a subset of B; denoted as $A \subset B$. in particular $A \subset A$

Remark. $A \subset B$ is sometimes written as $A \subseteq B$ to emphasize the fact that $A = B$ is a possibility. If $A \subset B$ but $A \neq B$, then A is said to be a *proper subset* (or a strict subset) of B, written as $A \Subset B$.

$A \subset B \Leftrightarrow B \supset A$: B is a *supset* of A.

$A \subset C$

Example 1.1.1.

1. $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$, $C = \{1, 2, 3, 4, 5\}$. Then $A \subseteq C$ (in fact $A \Subset C$), $1 \in A$, but $1 \notin B$ and $B \not\subseteq C$.
2. C = the set of all students studying at CUHK. M = the set of all math major students currently studying at CUHK. Then $M \subseteq C$. ~~You $\in C$~~.

Example 1.1.2. Some important number sets:

1. $\mathbb{N}$: the set of all natural numbers (positive integers) $= \{1, 2, 3, \ldots\}$.
2. $\mathbb{Z}$: the set of all integers $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. $= \{0, \pm 1, \pm 2, \cdots\}$
3. $\mathbb{Q}$: the set of all rational numbers $= \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.

$\mathbb{N} \subset \mathbb{Z}$

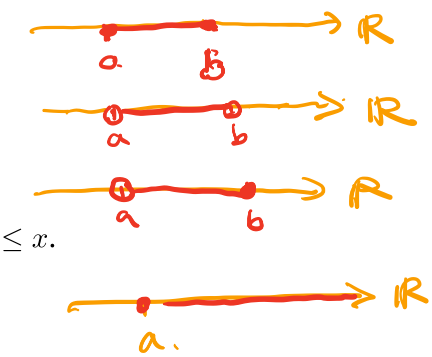
$\mathbb{Z} \subset \mathbb{Q}$

4. $\mathbb{R}$: the set of all real numbers.

Remark. If the elements in a set can be ordered and the ordering are taken into account in the definition, then it is called an *ordered set*. E.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may be viewed as ordered sets.

1.2 Intervals

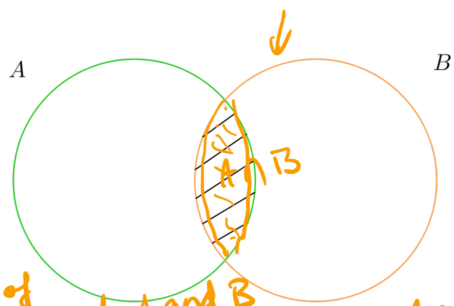
- $[a, b] = \{x \mid a \leq x \leq b\}$. (closed interval)
- $(a, b) = \{x \mid a < x < b\}$. (open interval)
- $(a, b] = \{x \mid a < x \leq b\}$.
- $[a, \infty)$: the set of all real numbers x such that $a \leq x$.



Drawing open/closed intervals on the real line:

1.3 Set operations

Let A, B be two sets:

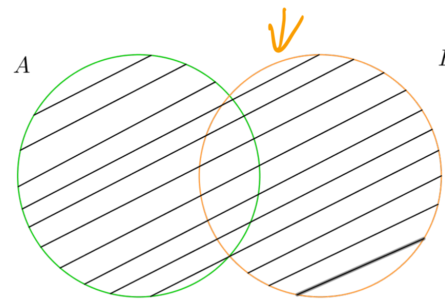


the set of intersection of A and B

Intersection

$A \cap B = \{x \,|\, x \in A$ and $x \in B\}$

A∩B ⊂ A
A∩B ⊂ B

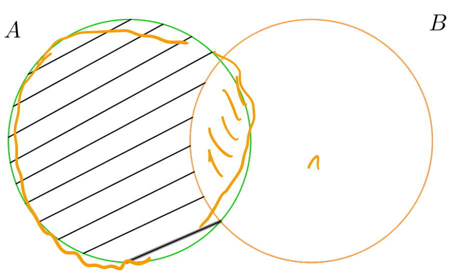


A ⊂ A∪B
B ⊂ A∪B

Union

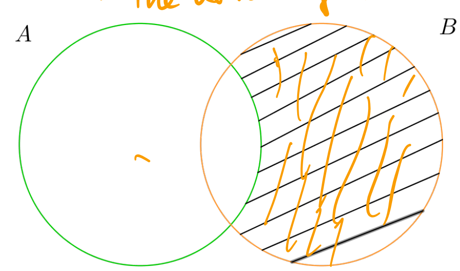
$A \cup B = \{x \,|\, x \in A$ or $x \in B\}$

the union of A and B



Relative complement of B in A

A-B = $A \backslash B = \{x \,|\, x \in A$ and $x \notin B\}$



Relative complement of A in B

$B \backslash A = \{x \,|\, x \in B$ and $x \notin A\}$ = B-A

Remark. Alternate notation for $A\backslash B$: $A - B$.

Example 1.3.1.

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5\}$.
   $A \cap B = \{2, 3\}$, $A \cup B = \{1, 2, 3, 4\}$, $A\backslash B = \{1\}$, $B\backslash A = \{4\}$, $A\backslash C = A$. = {1, 2, 3}
2. $\mathbb{R}\backslash\{a\}$: the set of all real numbers x, except $x = a$. = $(-\infty, a) \cup (a, \infty)$ a   ℝ
3. $A\backslash B = A\backslash(A \cap B)$.

Exercise 1.3.1.

1. What are the meanings of the following sets
   (a) $(-\infty, a)$.   a $= \mathbb{R} \backslash [a, \infty)$
   (b) $\mathbb{R}\backslash\{1, 2, 3\}$ 1 2 3 ℝ $= (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, \infty)$
   (c) $\mathbb{R}\backslash[2, 3)$.   2 3 $= (-\infty, 2) \cup [3, \infty)$ $[2, 3)$
2. Show that $\mathbb{R}\backslash[1, \infty) = (-\infty, 1)$.

domain of f codomain of f

E.g.: $A = \{1,2\}$ $B = \{3,1\}$ Range of f

$f: A \to B$ $f(1) = 3$ $f(2) = 3$ $= \{f(1), f(2)\} = \{3\}$

1.4 Functions

Definition 1.4.1. A **function** is a rule that assigns to EACH element x in a set A EXACTLY ONE element y in a set B. If the function is denoted by f, then we may write

$$f : A \to B.$$

The set A is called the domain of the function. The set B is called the codomain of f. The assigned elements in B is called the range of f.

set of all

$x \in A$ is the independent variable of f; $y = f(x) \in B$ is the dependent variable of f.

Given $a \in A$, $f(a) \in B$ is said to be the *value* of the function f at a. Given $S \subset A$,

$$f(S) := \{f(a) \,|\, a \in S\} \quad \subset B$$

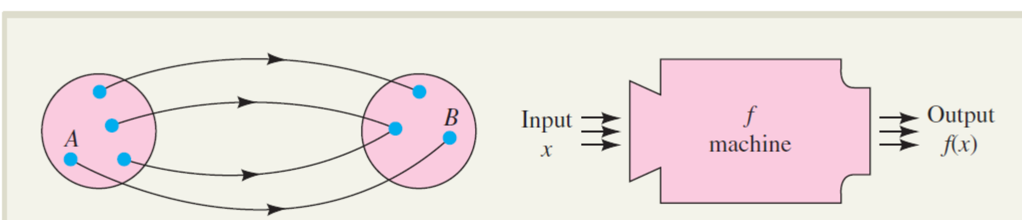
is said to be the *image* of S under f. In particular, the "range" of f, as defined above, is $f(A) \subset B$.

When the domain and range of a function are both sets of real numbers, the function is said to be a real-valued function of one variable, and we write

$$f : \overset{x}{\mathbb{R}} \to \mathbb{R}.$$

Most functions encountered in this course are real-valued functions of one variable. Unless otherwise specified, a function is a real-valued function of one variable in this course.

Remark. There is some ambiguity in the definition of "range" in math literature. See the Wiki article. A function $f : A \to B$ is also called a *map from A to B*; A is the *source* of f and B is the *target* of f.



(a) A function as a mapping (b) A function as a machine

Example 1.4.1. $f : [-1, 3) \to \mathbb{R}$ is defined by $f(\overset{0}{x}) = x^2 + 4$ (sometimes written as $y = x^2 + 4$). Then

$$f(0) = (0)^2 + 4 = 4.$$

domain $= [-1, 3)$, codomain $= \mathbb{R}$, range of f $= [4, 13)$.

$= \{y \mid y = x^2+4,\ -1 \le x < 3\}$

$= [4,\ 9+4=13) \subset \mathbb{R}$

Remark. If a function is given by a formula without domain specified, then assume domain = set of all x for which $f(x)$ is well defined, this domain is also called the natural domain of f.

Example 1.4.2. Find the natural domain of the functions.

1. $f(x) = \dfrac{1}{x-3}$. ↙ doesn't make sense when $x=3$, natural domain $= \mathbb{R}\setminus\{3\}$
2. $g(t) = \dfrac{\sqrt{3-2t}}{t^2+4}$.

denominator $t^2+4>0$

$\sqrt{3-2t}$ is defined only when $3-2t \geq 0$ $\Updownarrow$ $3/2 \geq t$

Solution.

1. $\dfrac{1}{x-3}$ is not defined when its denominator $x-3=0$, i.e. $x=3$. So the domain is $\mathbb{R}\setminus\{3\}$.
2. The domain of $\sqrt{3-2t}$ consists of all x such that $3-2t \geq 0$, which implies that $t \leq \frac{3}{2}$. Hence the natural domain is $\boxed{(-\infty, \frac{3}{2}]}$.

■

$x^2-1=(x+1)(x-1)$

Example 1.4.3. Let $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x+1$. Can we say f and g are the same function?

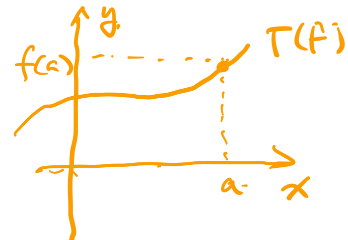
Solution. No! The domain of $f(x)$ is $\mathbb{R}\setminus\{1\}$, the domain of $g(x)$ is $\mathbb{R}$. Only when $x \neq 1$, $f(x) = g(x)$. ■

1.4.1 Vertical Line Test for Graph

$f: \mathbb{R} \to \mathbb{R}$

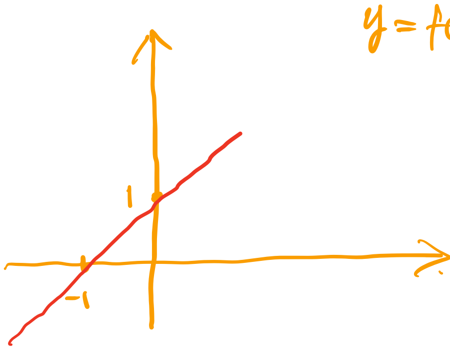
A way to visualize a function is its graph. If f is a real-valued function of one variable, its graph consists of the points in the Cartesian plane $\mathbb{R}^2$ whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\Gamma(f) := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y = f(x)\}.$$



Review: Graphing a real-valued function of one variable: [HBSP] 1.2.

Example 1.4.4. linear functions; piecewise linear functions; quadractic functions, exponential and log functions, trig functions.

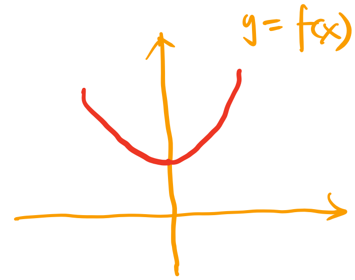


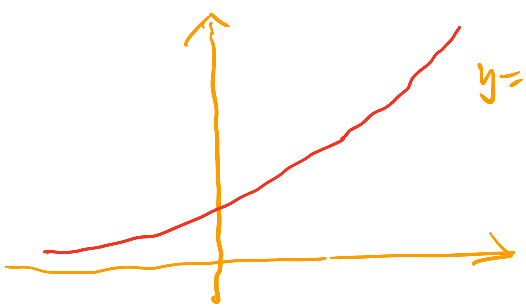
$y = f(x) = x+1$

a linear function
$ax+b$
$a, b \in \mathbb{R}$
fixed

(graph is a straight line)

$y = f(x) = x^2+1$

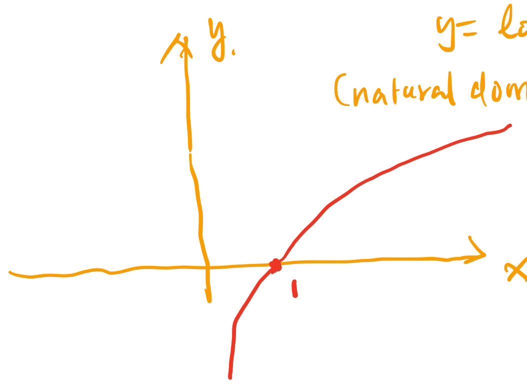




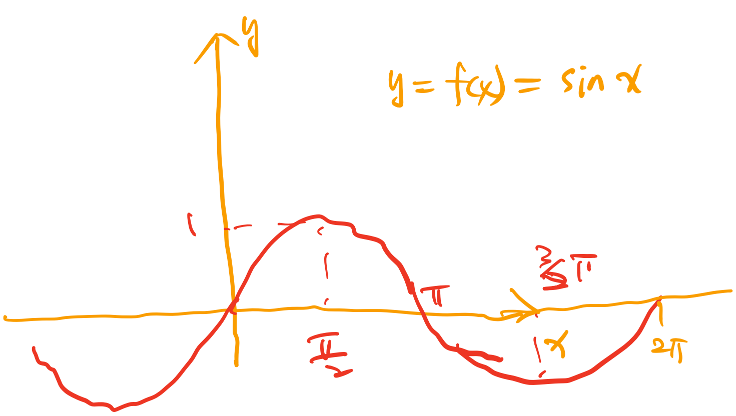
$y = f(x) = e^x$

$y = \log x$

(natural domain $(0, \infty)$)

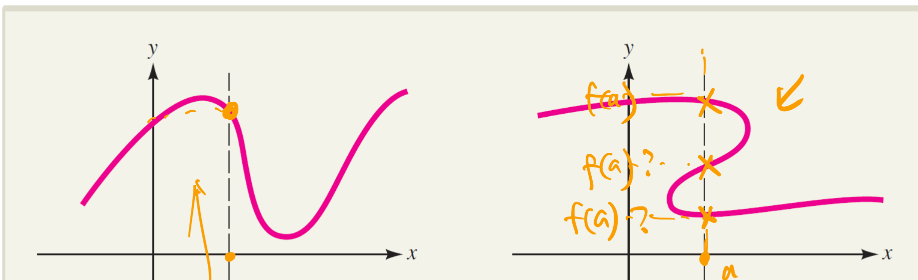


$y = f(x) = \sin x$



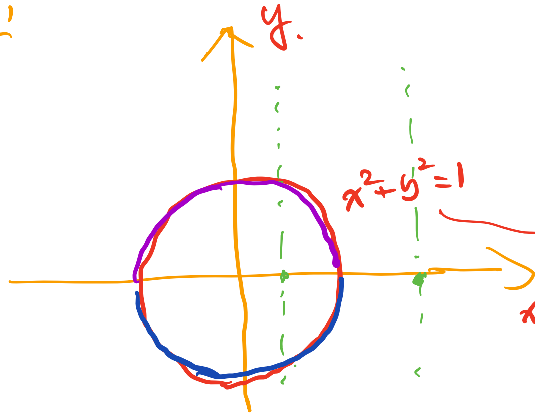
It is important to realize that not every curve is the graph of a function. For instance, suppose the circle $x^2 + y^2 = 5$ were the graph of some function $y = f(x)$. Then, since the points $(1, 2)$ and $(1, -2)$ both lie on the circle, we would have $f(1) = 2$ and $f(1) = -2$, contrary to the requirement that a function assigns one and only one value to each number in its domain. Geometrically, this happens because the vertical line $x = 1$ intersects the graph of the circle more than once. The vertical line test is a geometric rule for determining whether a curve is the graph of a function.

The Vertical Line Test A curve is the graph of a function if and only if no vertical line intersects the curve more than once:



(a) The graph of a function (b) Not the graph of a function

Es,



is not the graph of a function

$y = \pm\sqrt{1-x^2}$

$y = \sqrt{1-x^2}$

functions $y = -\sqrt{1-x^2}$

natural domain $[-1, 1]$

1.4.2 Some Special Functions

Definition 1.4.2. A piecewise function is defined by more than one formula, with each individual formula defined on a subset of the domain.

Example 1.4.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

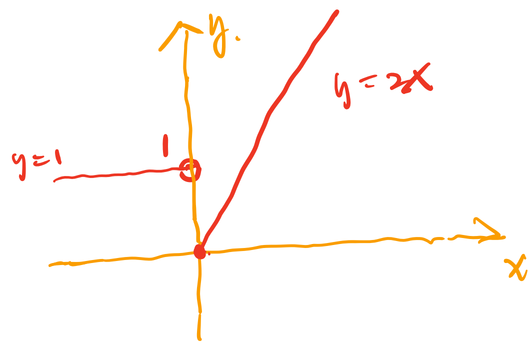
use 1st formula

2nd formula

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2x, & \text{if } x \geq 0. \end{cases}$$

Then $f(-1) = 1$, $f(0) = 0$ and $f(1) = 2$. → 2nd formula.

Remark. If all the formulae involved in defining a piecewise function are linear, then the function is said to be *piecewise linear*. E.g. The function in the preceding example.



Example 1.4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & \text{if } |x| \geq \pi \\ \sin x, & \text{if } |x| < \pi. \end{cases}$$